

# CONTINUITY OF CP-SEMIGROUPS IN THE POINT-STRONG OPERATOR TOPOLOGY

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**ABSTRACT.** We prove that if  $\{\phi_t\}_{t \geq 0}$  is a CP-semigroup acting on a von Neumann algebra  $M \subseteq B(H)$ , then for every  $A \in M$  and  $\xi \in H$ , the map  $t \mapsto \phi_t(A)\xi$  is norm-continuous. We discuss the implications of this fact to the existence of dilations of CP-semigroups to semigroups of endomorphisms.

## 1. INTRODUCTION

Let  $H$  be a Hilbert space, not necessarily separable, and let  $M \subseteq B(H)$  be a von Neumann algebra. A *CP-semigroup* on  $M$  is a family  $\phi = \{\phi_t : M \rightarrow M\}_{t \geq 0}$  of contractive normal completely positive maps which satisfies the following properties:

- (1)  $\phi_0(A) = A, \forall A \in M$
- (2)  $\phi_{s+t} = \phi_s \circ \phi_t, s, t \geq 0$
- (3) for all  $A \in M$  and  $\omega \in M_*$ ,  $\lim_{t \rightarrow t_0} \omega(\phi_t(A)) = \omega(\phi_{t_0}(A))$

where  $M_*$  denotes the predual of  $M$ . We shall refer to continuity condition (3) as *continuity in the point- $\sigma$ -weak topology*. It is equivalent to *continuity in the point-weak operator topology*, i.e.

$$\lim_{t \rightarrow t_0} \langle \phi_t(A)\xi, \eta \rangle = \langle \phi_{t_0}(A)\xi, \eta \rangle, \quad A \in M, \xi, \eta \in H.$$

A CP-semigroup  $\phi$  is called an *E-semigroup* if  $\phi_t$  is a  $*$ -endomorphism for all  $t \geq 0$ .

In this note we prove that CP-semigroups satisfy a seemingly stronger continuity condition, namely

$$(1.1) \quad \lim_{t \rightarrow t_0} \|\phi_t(A)\xi - \phi_{t_0}(A)\xi\| = 0,$$

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for all  $A \in M, \xi \in H$ . A semigroup satisfying (1.1) will be said to be *continuous in the point-strong operator topology*. The proposition that CP-semigroups are continuous in the point-strong operator topology has appeared in the literature earlier, but the proofs that are available seem to be incomplete. In the proofs of which we are aware, only continuity *from the right* in the point-strong operator topology is established. By this we mean that (1.1) holds for limits taken with  $t \searrow t_0$ .

We consider the continuity of CP-semigroups in the point-strong operator topology to be an important property, because it plays a crucial role in the existence of dilations of CP-semigroups to E-semigroups. We are aware of five different proofs for the fact that every CP-semigroup has a dilation to an E-semigroup: Bhat [2], Selegue [7], Bhat–Skeide [4], Muhly–Solel [6] and Arveson [1] (some of the authors require some additional conditions, notably that the CP-semigroup be unital or that the Hilbert space be separable). In order to show that the minimal dilation of a CP-semigroup to an E-semigroup is continuous in the point-weak operator topology, all authors make use of continuity of the CP-semigroup in the point-strong operator topology, either implicitly or explicitly.

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## 2. PRELIMINARIES

Let  $M$  be a von Neumann algebra acting on a Hilbert space, which is not assumed to be separable. Let  $\phi = \{\phi_t : M \rightarrow M\}_{t \geq 0}$  be a CP-semigroup acting on  $M$ . We denote by  $M_*$  the set of  $\sigma$ -weakly continuous linear functionals on  $M$ . We shall denote by  $\sigma(M_*, M)$  the pointwise convergence topology of  $M_*$  as a subset of the dual space of  $M$ .

Let  $\delta$  be the generator of  $\phi$ , and let  $D(\delta)$  denote its domain:

$$D(\delta) = \{A \in M : \exists \delta(A) \in M \forall \omega \in M_* \lim_{t \rightarrow 0+} t^{-1} \omega(\phi_t(A) - A) = \omega(\delta(A))\}.$$

**Lemma 2.1.** *For every  $A \in M$  and  $\xi \in H$ , the map  $t \mapsto \phi_t(A)\xi$  is continuous from the right (in norm).*

The proof of this result can be found in the literature, for example as Lemma A.1 of [3] or Proposition 4.1 item 1 in [6]. For completeness, let us present the argument from [3]. Let  $A \in M$ ,  $\xi \in H$  and  $t \geq 0$ .

For all  $h > 0$ , we have, using the Schwartz inequality for completely positive maps,

$$\begin{aligned} & \|\phi_{t+h}(A)\xi - \phi_t(A)\xi\| = \\ &= \langle \phi_{t+h}(A)^* \phi_{t+h}(A)\xi, \xi \rangle - 2 \operatorname{Re} \langle \phi_{t+h}(A)\xi, \phi_t(A)\xi \rangle + \|\phi_t(A)\xi\|^2 \\ &\leq \langle \phi_h(\phi_t(A)^* \phi_t(A))\xi, \xi \rangle - 2 \operatorname{Re} \langle \phi_{t+h}(A)\xi, \phi_t(A)\xi \rangle + \|\phi_t(A)\xi\|^2 \xrightarrow{h \rightarrow 0} 0. \end{aligned}$$

We remark, however, that two-sided continuity does not follow directly from continuity from the right. This is in contrast with the situation of the classical theory of  $C_0$ -semigroups on Banach spaces (see for example [5]). If  $T = \{T_t\}_{t \geq 0}$  is a semigroup of contractions on a Banach space  $X$  such that the maps

$$t \mapsto T_t(x)$$

are continuous from the right in norm for all  $x \in X$ , then it is easy to show that these maps are also continuous from the left in norm<sup>1</sup>. In fact, when  $X$  is separable, for instance, it can be proven by measurability and integrability techniques that if the maps  $t \mapsto f(T_t(x))$  are measurable for all  $x \in X$  and  $f \in X^*$ , then the maps  $t \mapsto T_t(x)$  are continuous in norm for  $t > 0$ . In the case of CP-semigroups on von Neumann algebras, however, these techniques seem to require considerable modification. We provide here an alternative approach to the problem.

Recall that a function  $g : [0, 1] \rightarrow H$  is *weakly measurable* if for all  $\eta \in H$ , the complex-valued function  $g_\eta(t) = \langle \eta, g(t) \rangle$  is measurable. We will say that the function  $g$  is *strongly measurable* if there exists a family of countably-valued functions (i.e. assuming a set of values which is at most countable) converging Lebesgue almost everywhere to  $g$ . (For more details, see Definition 3.5.4, p. 72, and the surrounding discussion in [5]).

**Lemma 2.2.** *For all  $\xi \in H$ ,  $A \in M$ , the function  $f : [0, 1] \rightarrow H$  given by  $f(t) = \phi_t(A)\xi$  is strongly measurable and Bochner integrable on the interval  $[0, 1]$ .*

*Proof.* The function  $f$  is weakly continuous, since  $\phi$  is continuous in the point-weak operator topology. In particular, it is weakly measurable. Furthermore, by Lemma 2.1, the function  $f$  is continuous from the right in norm, hence it is separably valued (i.e., its range is contained in a separable subspace of  $H$ ). By Theorem 3.5.3 of [5], the function  $f$

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<sup>1</sup> for given  $x \in X$ ,  $0 \leq t \leq a$ ,  $\|T_{a-t}(x) - T_a(x)\| = \|T_{a-t}(x - T_t(x))\| \leq \|x - T_t(x)\|$ .

is strongly measurable because it is weakly measurable and separably valued.

Thanks to Theorem 3.7.4, p. 80 of [5], in order to show that  $f$  is Bochner integrable it is enough to show that  $f$  is strongly measurable and that

$$\int_0^1 \|f(t)\| dt < \infty.$$

The latter condition is easy to verify, as  $t \mapsto \|f(t)\|$  is a right-continuous, bounded function on  $[0, 1]$ .  $\square$

We thank Michael Skeide for the idea to use the continuity of  $f$  from the right in order to avoid making the assumption that  $H$  is separable.

**Lemma 2.3.** *Let  $A \in B(H)$  be positive. Then there exists a sequence  $A_n \in D(\delta)$  of positive operators such that  $A_n \rightarrow A$  in the  $\sigma$ -strong\* topology.*

*Proof.* Recall that the sequence

$$A_n = n \int_0^{1/n} \phi_t(A) dt$$

(integral taken in the  $\sigma$ -weak sense) converges in the  $\sigma$ -weak topology to  $A$ . Furthermore  $A_n \in D(\delta)$  and it is a positive operator for each  $n > 0$  since  $\phi_t$  is a CP map for all  $t$ . It is easy to check that  $\|A_n\| \leq \|A\|$  for all  $n$  since  $\phi_t$  is contractive.

Now observe that for each  $\xi \in H$ , the map  $t \mapsto \phi_t(A)\xi$  is Bochner integrable on  $[0, 1]$  (see Lemma 2.2), hence in fact we have

$$A_n \xi = n \int_0^{1/n} \phi_t(A) \xi dt$$

where the integral is taken in the Bochner sense. The identity holds because for all  $\eta \in H$ ,  $n \in \mathbb{N}$  we have:

$$\langle A_n \xi, \eta \rangle = n \int_0^{1/n} \langle \phi_t(A) \xi, \eta \rangle dt = \langle n \int_0^{1/n} \phi_t(A) \xi dt, \eta \rangle.$$

We now show that  $A_n \rightarrow A$  strongly. Let  $\xi \in H$  be fixed.

$$\begin{aligned} \|A\xi - A_n\xi\| &= \|n \int_0^{1/n} A\xi dt - n \int_0^{1/n} \phi_t(A)\xi dt\| \\ &\leq n \int_0^{1/n} \|A\xi - \phi_t(A)\xi\| dt. \end{aligned}$$

The latter goes to zero by continuity from the right (Lemma 2.1). Since  $A_n, A$  are positive operators, by considering adjoints we obtain

that  $A_n \rightarrow A$  in the strong\* topology. Finally, since the sequence is bounded, we have convergence in the  $\sigma$ -strong\* topology.  $\square$

**Lemma 2.4.** *Let  $A_n$  be a bounded sequence of operators in  $M$  converging to  $A$  in the  $\sigma$ -strong\* topology and let  $t_0 \geq 0$ . Then for every sequence  $t_k \rightarrow t_0$ ,  $\xi \in H$  and  $\epsilon > 0$ , there exists  $N \in \mathbb{N}$  such that for  $n \geq N$ ,*

$$\|\phi_{t_k}(A_n - A)\xi\| < \epsilon, \quad \text{for all } k.$$

*Proof.* Let  $B_n = (A_n - A)^*(A_n - A)$ ,  $\omega_k(X) = \langle \phi_{t_k}(X)\xi, \xi \rangle$  and  $\omega(X) = \langle \phi_{t_0}(X)\xi, \xi \rangle$ . Then we have that

$$\|\phi_{t_k}(A_n - A)\xi\|^2 = \langle \phi_{t_k}(A_n - A)^* \phi_{t_k}(A_n - A)\xi, \xi \rangle \leq \omega_k(B_n)$$

since  $\phi_t$  is a CP map for all  $t$ . Since  $\phi$  is a point- $\sigma$ -weakly continuous semigroup, we have that  $(\omega_k)$  is a sequence of  $\sigma$ -weakly continuous linear functionals such that  $\omega_k(X) \rightarrow \omega(X)$  for all  $X \in M$ . Furthermore,  $B_n$  is a bounded sequence converging in the  $\sigma$ -strong\* topology to 0. The latter holds because  $A_n$  is a bounded sequence converging to  $A$  in the  $\sigma$ -strong\* topology and multiplication is jointly continuous with respect to this topology in bounded sets (of course \* is also continuous). Finally, we obtain the desired conclusion by applying Lemma III.5.5, p.151 of [8], which states the following. Let  $M$  be a von Neumann algebra and let  $\rho_k$  be a sequence in  $M_*$  converging to  $\rho_0 \in M_*$  in the  $\sigma(M_*, M)$  topology. If a bounded sequence  $(a_n)$  converges  $\sigma$ -strongly\* to 0, then  $\lim_{n \rightarrow \infty} \rho_k(a_n) = 0$  uniformly in  $k$ .  $\square$

### 3. THE MAIN RESULT

**Theorem 3.1.** *Let  $\phi$  be a CP-semigroup acting on a von Neumann algebra  $M \subseteq B(H)$ . Then for all  $\xi \in H$ ,  $A \in M$  and  $t_0 \geq 0$ ,*

$$\lim_{t \rightarrow t_0} \|\phi_t(A)\xi - \phi_{t_0}(A)\xi\| = 0.$$

*Proof.* Let  $\epsilon > 0$  be given, and let  $(t_k)$  be a sequence converging to  $t_0$ . By Lemma 2.3, there is a bounded sequence  $(A_n)$  of operators  $A_n \in D(\delta)$  such that  $A_n \rightarrow A$  in the  $\sigma$ -strong\* topology. By Lemma 2.4, there exists  $N \in \mathbb{N}$  such that for  $n \geq N$ ,

$$\|\phi_{t_k}(A_n - A)\xi\| < \frac{\epsilon}{3}, \quad \text{for all } k \geq 0.$$

By an application of the Principle of Uniform Boundedness, if  $X \in D(\delta)$  there exists  $C_X > 0$  such that

$$\sup_{s > 0} \frac{1}{s} \|\phi_s(X) - X\| \leq C_X < \infty.$$

Now notice that  $A_n \in D(\delta)$  for all  $n$ , and in particular  $\exists C > 0$  such that

$$\sup_{s>0} \frac{1}{s} \|\phi_s(A_N) - A_N\| \leq C.$$

Because  $\phi_s$  is a contraction for all  $s$ , we obtain that for all  $k$ ,

$$\begin{aligned} \|\phi_{t_k}(A_N)\xi - \phi_{t_0}(A_N)\xi\| &\leq \|\phi_{t_k}(A_N) - \phi_{t_0}(A_N)\| \|\xi\| \\ &\leq \|\phi_{|t_k-t_0|}(A_N) - A_N\| \|\xi\| \\ &\leq C\|\xi\| |t_k - t_0|. \end{aligned}$$

In particular, we must have that  $\|\phi_{t_k}(A_N)\xi - \phi_{t_0}(A_N)\xi\| \rightarrow 0$  as  $k \rightarrow \infty$ . Thus there is  $K \in \mathbb{N}$  such that for  $k \geq K$ ,

$$\|\phi_{t_k}(A_N)\xi - \phi_{t_0}(A_N)\xi\| < \frac{\epsilon}{3}.$$

We conclude that for  $k \geq K$ ,

$$\begin{aligned} \|\phi_{t_k}(A)\xi - \phi_{t_0}(A)\xi\| &\leq \|\phi_{t_k}(A - A_N)\xi\| + \\ &+ \|\phi_{t_k}(A_N)\xi - \phi_{t_0}(A_N)\xi\| + \|\phi_{t_0}(A_N - A)\xi\| < \epsilon. \end{aligned}$$

□

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